AN ESTIMATE OF THE MAXIMAL OPERATORS ASSOCIATED WITH GENERALIZED LACUNARY SETS

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Abstract. Let Ω be any set of directions (unit vectors) on the plane. Denote by $\mathcal{R}_\Omega$ the set of all rectangles which have a side parallel to some direction from Ω. In this paper we study maximal operators on the plane $\mathbb{R}^2$ defined by

$$M_\Omega f(x) = \sup_{x\in R\in\mathcal{R}_\Omega} \frac{1}{|R|}\int_R |f(y)|dy.$$

We are interested in extensions of lacunary sets of directions, to collections we call N–lacunary, for integers N. We proceed by induction. Say that $\Omega = \{v_k \mid k\in\mathbb{N}\}$ is 1–lacunary iff for each integer k, v_k and v_{k+1} are neighboring points, and there is a direction v_∞ so that

$$\tfrac{1}{2}|v_k - v_{k+1}| < |v_{k+1} - v_\infty| < |v_k - v_{k+1}|.$$

Every $N+1$–lacunary set can be obtained from some N–lacunary Ω_N adding some points to Ω_N. Between each two neighbor points $a, b \in \Omega_N$ we can add a 1–lacunary sequence (finite or infinite). We show that for all N lacunary sets Ω,

$$\|M_\Omega f(x)\|_2 \lesssim N\|f\|_2.$$

Observe that every set Ω of N points is $(C\log N)$–lacunary. We then obtain a Theorem of N. Katz [18]. Both the current inequality, and Katz' result are consequence of a general result of Alfonseca, Soria, and Vargas [3]. We offer the current proof as a succinct, self–contained approach to this inequality.

1. Introduction

Let Ω be any set of directions (unit vectors) on the plane. Denote by $\mathcal{R}_\Omega$ the set of all rectangles which have a side parallel to some direction from Ω. In this paper we study maximal operators on the plane $\mathbb{R}^2$ defined by

$$M_\Omega f(x) = \sup_{x\in R\in\mathcal{R}_\Omega} \frac{1}{|R|}\int_R |f(y)|dy. \tag{1.1}$$

A. Nagel, E.M. Stein and S. Wainger [19] using Fourier transform method proved the boundedness of $M_\Omega f(x)$ in spaces L^p, $1 < p < \infty$ for any lacunary set of directions $\Omega = \{\theta_k\}$, $(\arg\theta_{k+1} < \lambda\arg\theta_k,\ \lambda < 1)$.

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Every $N+1$–lacunary set can be obtained from some N–lacunary Ω_N adding some points to Ω_N. Between each two neighbor points $a, b \in \Omega_N$ we can add a 1–lacunary sequence (finite or infinite). So if Ω is some N–lacunary set we can fix a sequence of sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_{N-1} \subset \Omega$ such that each Ω_k is k–lacunary.

It is commonly known that maximal functions in N–lacunary directions are bounded for all integers N. For instance, the case of 2–lacunary is due to P. Sjögren and P. Sjölin [20]. We are interested in growth of the norm of M_Ω for N–lacunary, as N tends to infinity.

Theorem 1. *For all integers N, and all N–lacunary sets Ω we have*

$$\|M_\Omega f(x)\|_2 \lesssim N\|f\|_2.$$

It is easy to check that each set of directions of cardinality N is $(C \log N)$–lacunary, for an absolute constant C. Therefore, as a corollary, we see that for finite collections Ω, we have

$$\|M_\Omega f\|_2 \lesssim (\log \sharp\Omega)\|f\|_2. \tag{1.2}$$

This inequality is due to N. Katz [18]. This estimate is sharp as the power of $(\log \sharp\Omega)$, and so in the Theorem, our estimate is sharp as to the power of N.

Both Katz' result and our Theorem is a consequence of a more general result of Alfonseca, Soria, and Vargas [3], a result we recall in more detail below. The current proof is succinct, and self–contained, and so may prove to be of some independent interest.

We close this section with a more detailed, but far from complete, description of the history of this question, and the relationship of our result to the literature. In 1977, A. Cordoba [7] considered the maximal function formed over all rectangles that are 1 by N, obtaining a slow increase in the norm on L^2. Thus, the set Ω is uniformly distributed, but one only considers rectangles of one aspect ratio. The method of proof employed a geometric method to prove a covering lemma. The method, as described in A. Cordoba and R. Fefferman [9], was broadly influential. The point of view adopted in this paper was formalized in an article from 1979 by S. Wainger [24]. The estimate (1.2) in the instance of uniformly distributed directions was proved by J. Stromberg [22], in 1978.

On the other hand, there were natural reasons to expect that the instance of lacunary directions would behave differently, and was investigated by J. Stromberg [21]. The full range of L^p, $1 < p < \infty$, inequalities in this instance was established by Fourier analysis, and square function methods by A. Nagel, S. Wainger, and E.M. Stein [19], a method that also proved to be influential. These results are related to interesting results on multipliers, as shown by A. Cordoba and R. Fefferman [10]. For extensions of this, see A. Carbery [6].

An interesting question was if Stromberg's result [22] in the uniformly distributed case extended to the case of N distinct directions. A partial result was treated by Barrionuevo [4, 5]. And the definitive result was obtained by N. Katz [18]. His method of proof is a clever duality argument, relying on an John–Nirenberg type to obtain the required estimate.

At this point, we note that there is a distinction between the case of rectangles of all aspect ratios, as we do, and the case of a fixed aspect ratio. It is the later case that is considered by e.g. A. Cordoba [7], and in Katz' paper [17].

An interesting question concerns the maximal function computed in a set of directions specified by a Cantor set of directions. For the ordinary middle third Cantor set, there is a partial result on L^2 by A. Vargas [23]. Yet, this full maximal function is unbounded on L^2, as proved by N. Katz [16]. It would be interesting to obtain meaningful information about this maximal operator on L^p, for $p > 2$. K. Hare [13] uses Katz' argument, with more general Cantor sets.

Recently, A. Alfonesca, F. Soria and A. Vargas [2, 3], also see Alfonseca [1], have proved an interesting orthogonality principle for these maximal functions. Let $\Omega = \{v_k \mid k \in \mathbb{N}\}$ be a set of directions, and between two neighboring directions v_k, v_{k+1}, let Ω_k be an arbitrary set of directions. Then, ([3]) it is the case that

$$\|M\|_{2\to 2} \le C\|M_\Omega\|_{2\to 2} + \sup_k \|M_{\Omega_k}\|_{2\to 2}.$$

What is essential is that the second term occurs with constant 1. This proves our Theorem. Let $\eta(N)$ be the maximum of $\|M_{\Omega_N}\|_{2\to 2}$, with the maximum taken over all N–lacunary sets of directions. The inequality above clearly implies that $\eta(N) \le C\eta(1) + \eta(N-1)$. Iterating the inequality $N-1$ times proves the Theorem.

General necessary and sufficient conditions on Ω for the boundedness of M_Ω have been sought by J. Duoandikoetxea, and A. Vargas [11], with extensions by K. Hare, and J. Rönning [14, 15].

A paper by M. Christ [8] includes examples of sets of directions Ω, and partial results on the norm boundedness of M_Ω which are not incorporated into the theories associated with this subject. K. Hare and F. Ricci [12] have established an interesting variant of the lacunary directional maximal function.

2. Notations

By $A \lesssim B$ we mean that there is an absolute constant K so that $A \le KB$. By $\widehat{f}(\xi)$, we mean the Fourier transform of f, thus

$$\widehat{f}(\xi) = \int f(x)e^{ix\cdot\xi}\,dx$$

We use a well–known reduction to parallelograms. It is clear that we can associate directions in Ω to points in e.g. $(0, 1/4)$. Denote

$$P_\alpha f(x) = \sup_{\delta_1,\delta_2} \frac{1}{4\delta_1\delta_2} \int_{x_1-\delta_1}^{x_1+\delta_1} \int_{x_2-x_1\alpha-\delta_2}^{x_2-x_1\alpha+\delta_2} |f(t_1, t_2)|\, dt_2 dt_1. \tag{2.1}$$

This is a maximal function over parallelograms, with one side parallel to the x axis, and the other side forming an angle of slope α with the x axis. Then in order to prove the theorem it is sufficient to prove

$$\| \sup_{\alpha\in\Omega} P_\alpha f\|_2 \le CN\|f\|_2$$

where Ω is any N–lacunary set from $(0, 1)$.

Our method of proof is Fourier analytic, and we shall find it convenient to use the the Fejer kernel

$$K_r(x) = \int_{-r}^{r} \left(1 - \frac{|t|}{r}\right) e^{-itx} dt = \frac{4\sin^2 \frac{Nx}{2}}{Nx^2}$$

For any r, R with $0 \le r < R/2$ we define the following functions

$$\psi_r(x) = 2K_{2r}(x) - K_r(x), \quad \psi_{r,R}(x) = \psi_R(x) - \psi_r(x)$$

Sometimes we will write $\psi_{0,r}$ instead of $\psi_r(x)$. We have

$$\widehat{\psi}_{r,R}(\xi) = \begin{cases} 1 & \text{if} & |\xi| \in [2r, R] \\ 0 & \text{if} & 0 \le |\xi| \le r \text{ or } |\xi| > 2R \\ \text{linear} & \text{on each} & \pm[r, 2r], \pm[R, 2R] \end{cases} \tag{2.2}$$

From a property of Fejer kernel we have

$$|\psi_{r,R}(x)| \le C \left(\max\left\{\frac{1}{Rx^2}, R\right\} + \max\left\{\frac{1}{rx^2}, r\right\}\right)$$

Thus for some sequence of intervals $\omega_k = \omega_{k,r,R}$ with centers at 0.

$$\begin{gathered} |\psi_{r,R}(x)| \le C \sum_k \gamma_k \frac{\mathbb{I}_{\omega_k}(x)}{|\omega_k|} = \zeta_{r,R}(x) \\ \gamma_k > 0, \qquad \sum_k \gamma_k < 1, \qquad \omega_k \supset (1/R, 1/R). \end{gathered} \tag{2.3}$$

Choose a Schwartz function ϕ with

$$\phi \ge 0, \quad \operatorname{supp} \widehat{\phi} \subset [-1, 1]. \tag{2.4}$$

We can fix an even function λ with

$$\max\{|\phi(x)|, |x\phi(x)|\} \le \lambda(x), \qquad \int_{\mathbb{R}} \lambda(x)dx \le C, \tag{2.5}$$

Then define a Fourier analog of the average over parallelograms by

$$\Gamma^{\alpha}_{r,R,h}f(x) = \big(\psi_{r,R}(x_2 - x_1\alpha)\phi_h(x_1)\big) * f(x), \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.6}$$

where

$$\phi_h(x) = \frac{1}{h}\phi\left(\frac{x}{h}\right).$$

From (2.6) and (2.1) it follows that

$$P_\alpha f(x) \le C \sup_{R,h} \Gamma^{\alpha}_{R,h} f(x).$$

and therefore to prove our Theorem we need to verify the inequality

$$\| \sup_{R,h,\alpha\in\Omega} \Gamma^{\alpha}_{R,h} f(x)\|_2 \le CN\|f\|_2 \tag{2.7}$$

Taking the Fourier transform both sides of (2.6) we get

$$\widehat{\Gamma}^{\alpha}_{r,R,h}f(\xi) = \widehat{\phi}(h(\xi_1 + \xi_2\alpha))\widehat{\psi}_{r,R}(\xi_2)\widehat{f}(\xi) \tag{2.8}$$

3. Proof of Theorem

Lemma 1. *Let $\alpha, \beta \in (0,1)$ be any numbers and $0 < r < R, h > 0$. The operator $\Gamma^{\alpha}_{r,R,h}f(x)$ defined in (2.6) satisfies pointwise estimate*

$$|\Gamma^{\alpha}_{r,R,h}f(x)| \le C\,(hR|\alpha - \beta| + 1)\, P_\beta f(x), \quad x \in \mathbb{R}^2. \tag{3.1}$$

Proof. From (2.3) we have

$$\psi_{r,R}(x_2 - x_1\alpha) \le C\sum_k \frac{\gamma_k}{|\omega_k|}\mathbb{I}_{\omega_k}(x_2 - x_1\alpha)$$

where we have $|\omega_k| > 2/R$. Denote $\lambda(x_1) = 2Rx_1|\alpha - \beta| + 2$ and assume

$$x_2 - x_1\alpha \in \omega_k \tag{3.2}$$

for some k. Then taking account of (2.3) we get

$$\begin{aligned}\left|\frac{x_2 - x_1\beta}{\lambda(x_1)}\right| &= \left|\frac{x_2 - x_1\alpha + x_1(\alpha - \beta)}{\lambda(x_1)}\right| \\ &\le \left|\frac{x_2 - x_1\alpha}{2}\right| + \frac{1}{2R} \le \frac{|\omega_k|}{2},\end{aligned} \tag{3.3}$$

which means

$$\frac{x_2 - x_1\beta}{\lambda(x_1)} \in \omega_k. \tag{3.4}$$

Hence we conclude that (3.2) implies (3.4). Therefore

$$\mathbb{I}_{\omega_k}(x_2 - x_1\alpha) \le \mathbb{I}_{\omega_k}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right)$$

Finally we get

$$\psi_{r,R}(x_2 - x_1\alpha) \le C \sum_k \frac{\gamma_k}{|\omega_k|} \mathbb{I}_{\omega_k}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right) \le \zeta_{r,R}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right)$$

Thus taking account of (2.5) we obtain

$$\frac{1}{h}\phi\left(\frac{x_1}{h}\right)\zeta_{r,R}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right) \le C\left(hR|\alpha-\beta|+1\right)\frac{1}{h}\xi\left(\frac{x_1}{h}\right)\frac{1}{\lambda(x_1)}\zeta_{r,R}(\frac{x_2 - x_1\beta}{\lambda(x_1)})$$

from which we easily get (3.1). □

For any interval $J = (a, b)$ we denote by $S(J)$ the sector $\{ax_2 \le x_1 \le bx_2\}$. For any sector S define by $2S$ the sector which has same bisectrix with S and twice bigger angle. Denote by $T_S f$ the multiplier operator defined $\widehat{T}_S f = \mathbb{I}_S \widehat{f}$.

Lemma 2. *Let $J_1 \supset J_2 \supset \cdots \supset J_n$ be some sequence of intervals with*

$$J_k = [\alpha_k, \beta_k] \subset (0,1), \quad \operatorname{dist}\left((J_k)^c, J_{k+1}\right) \le |J_{k+1}|, \quad 1 \le k \le n \tag{3.5}$$

Then for any $\theta \in \bigcap J_k$ and any function $f \in L^2(\mathbb{R}^2)$ we have

$$\begin{aligned} P_\theta f &\lesssim P_0 f + P_\theta(T_{2S(J_n)} f) \\ &\quad + \sum_{k=1}^{n-1} P_{\alpha_k}(T_{2S(J_k)} f) + P_{\beta_k}(T_{2S(J_k)} f) \end{aligned} \tag{3.6}$$

where P_0 is a P_α with $\alpha = 0$.

Proof. Regard $\theta \in \bigcap J_k$ as fixed. For any R, h we have

$$\widehat{\Gamma}^\theta_{R,h} f(\xi) = \widehat{\psi}_R(\xi_2)\widehat{\phi}(h(\xi_1 + \xi_2\theta))\widehat{f}(x) \tag{3.7}$$

Denote

$$r_0 = 0, \quad r_k = \frac{2}{h|J_k|} \quad 1 \le k \le n. \tag{3.8}$$

From (2.2) it follows that

$$\widehat{\psi}_R(\xi_2) = \sum_{k=1}^m \widehat{\psi}_{2r_{k-1},r_k}(\xi_2) + \widehat{\psi}_{2r_m,R}(\xi_2) \tag{3.9}$$

where $m = \max\{k : r_k < 2R\}$. Denote

$$\begin{aligned} \Gamma_k f(x) &= \Gamma^\theta_{2r_k,r_{k+1},h} f(x) \quad 0 \le k < m, \\ \Gamma_m f(x) &= \Gamma^\theta_{2r_m,R,h} f(x). \end{aligned}$$

Then by (2.8) we have

$$\widehat{\Gamma}_k f(\xi) = \widehat{\psi}_{2r_{k-1},r_k}(\xi_2)\widehat{\phi}(h(\xi_1+\xi_2\theta))\widehat{f}(x) \quad 1 \le k < m$$
$$\widehat{\Gamma}_m f(x) = \widehat{\psi}_{2r_m,R}(\xi_2)\widehat{\phi}(h(\xi_1+\xi_2\theta))\widehat{f}(x)$$

and therefore using (3.9) we obtain

$$\Gamma^{\theta}_{R,h} f = \sum_{k=0}^{m} \Gamma_k f \tag{3.10}$$

Let us show

$$\begin{aligned} \operatorname{supp} \widehat{\psi}_{2r_k,r_{k+1}}(\xi_2)\widehat{\phi}(h(\xi_1+\xi_2\theta)) &\subset 2S(J_k), \quad 1 \le k < m, \\ \operatorname{supp} \widehat{\psi}_{2r_m,R}(\xi_2)\widehat{\phi}(h(\xi_1+\xi_2\theta)) &\subset 2S(J_m) \end{aligned} \tag{3.11}$$

From which it follows that

$$\Gamma_k f = \Gamma_k\big(T_{2S(J_k)} f\big), \quad 1 \le k \le m$$

Indeed, from (2.4) and (2.2) it follows that

$$\operatorname{supp} \widehat{\psi}_{2r_k,r_{k+1}}(\xi_2)\widehat{\phi}(h(\xi_1+\xi_2\theta))$$
$$= \{(\xi_1,\xi_2) : r_k \le \xi_2 \le 2r_{k+1},\ |\xi_1+\xi_2\theta| < \frac{1}{h}\}$$

The last set is a parallelogram with vertexes $(r_k\theta \pm \frac{1}{h}, r_k)$ and $(2r_{k+1}\theta \pm \frac{1}{h}, 2r_{k+1})$. These vertexes are from $2S(J_k)$ because

$$\frac{r_k\theta \pm \frac{1}{h}}{r_k} = \theta \pm \frac{|J_k|}{2}$$

which means $(r_k\theta \pm \frac{1}{h}, r_k) \in 2S(J_k)$. The same conclusion is true for next the pair of vertexes. This implies (3.11).

Using Lemma 1 we conclude

$$\begin{aligned} |\Gamma_k f| \lesssim{}& (hr_{k+1}\min\{|\theta-\alpha_k|, |\theta-\beta_k|\}+1)\times \\ & (P_{\alpha_k}\big(T_{2S(J_k)}f\big) + P_{\beta_k}\big(T_{2S(J_k)}f\big)) \quad 1 \le k < m \end{aligned} \tag{3.12}$$

Notice also

$$|\Gamma_0 f| \le P_0 f \tag{3.13}$$

$$|\Gamma_m f| \le P_\theta T_{2S(J_m)} f \tag{3.14}$$

By $\theta \in J_{k+1} \subset J_k$ and (3.5) we have

$$\min\{|\theta-\alpha_k|, |\theta-\beta_k|\} \le 2|J_{k+1}|$$

The last with (3.8) implies

$$hr_{k+1}\min\{|\theta-\alpha_k|, |\theta-\beta_k|\} \le 4$$

Hence by (3.12) we observe

$$|\Gamma_k f| \lesssim P_{\alpha_k}\big(T_{2S(J_k)}f\big) + P_{\beta_k}\big(T_{2S(J_k)}f\big), \quad 1 \le k < m.$$

Finally taking account also (3.13) and (3.14) we get Lemma 2.

□

Proof of Theorem 1. Let $\Omega \subset (0,1)$ be any N-lacunary set. We fix the sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_{N-1} \subset \Omega_N = \Omega$ from definition of N-lacunarity. Fix any angle $\theta \in \Omega$ and $R, h > 0$. Suppose

$$\theta \in \Omega_m \setminus \Omega_{m-1}, \text{ for some } m \le N. \tag{3.15}$$

Denote by G_k the set of all intervals whose vertexes are neighbor points in Ω_k. We can choose a sequence of intervals $J_k = [\alpha_k, \beta_k] \in G_k$ $k = 1, 2, \cdots, m$ such that

$$\theta \in \bigcap_{1 \le k \le m} J_k, \qquad \theta = \alpha_m \quad (\text{or } \theta = \beta_m)$$

It is clear that sequence J_k satisfies conditions of Lemma 2. Hence,

$$\begin{aligned} |M_\theta f|^2 &\lesssim \Big\{M_0 f + \sum_{k=1}^m (M_{\alpha_k}(T_{2S(J_k)}f) + M_{\beta_k}(T_{2S(J_k)}f))\Big\}^2 \\ &\lesssim |M_0 f|^2 + m \sum_{k=1}^m |M_\alpha(T_{2S(J)}f)|^2 + |M_\beta(T_{2S(J)}f)|^2 \end{aligned}$$

and therefore, summing over every interval $J = (\alpha, \beta) \in G_k$,

$$\sup_{\theta \in \Omega} |M_\theta f|^2 \lesssim |M_0 f|^2 + N \sum_{k=1}^N \sum_{J=(\alpha,\beta)\in G_k} |M_\alpha(T_{2S(J)}f)|^2 + |M_\beta(T_{2S(J)}f)|^2 \tag{3.16}$$

On the other hand using the $(2,2)$ bound of strong maximal operator we get for each $1 \le k \le N$,

$$\begin{aligned} \int_{\mathbb{R}^2} \sum_{J=(\alpha,\beta)\in G_k} |M_\alpha(T_{2S(J)}f)|^2 + |M_\beta(T_{2S(J)}f)|^2 dx &\lesssim \int_{\mathbb{R}^2} \sum_{J=(\alpha,\beta)\in G_k} \mathbb{I}_{2S(J)} |\widehat{f}|^2 d\xi \\ &\lesssim \int_{\mathbb{R}^2} |\widehat{f}|^2 d\xi \\ &= \int_{\mathbb{R}^2} |f|^2 dx \end{aligned}$$

Finally taking account of (3.16) we obtain

$$\int_{\mathbb{R}^2} \sup_{\theta\in\Omega} |M_\theta f|^2 dx \lesssim N^2 \int_{\mathbb{R}^2} |f|^2 dx$$

□

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